Euclidean Supergravity in Terms of Dirac Eigenvalues

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Abstract

It has been recently shown that the eigenvalues of the Dirac operator can be considered as dynamical variables of Euclidean gravity. The purpose of this paper is to explore the possibility that the eigenvalues of the Dirac operator might play the same role in the case of supergravity. It is shown that for this purpose some primary constraints on covariant phase space as well as secondary constraints on the eigenspinors must be imposed. The validity of primary constraints under covariant transport is further analyzed. It is shown that in this case restrictions on the tangent bundle and on the spinor bundle of spacetime arise. The form of these restrictions is determined under some simplifying assumptions. It is also shown that manifolds with flat curvature of tangent bundle and spinor bundle satisfy these restrictions and thus they support the Dirac eigenvalues as global observables.

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I. INTRODUCTION

Various attempts to understand the relationships between quantum theory and gravity have been made by many authors along the time. The approaches to this problem range from standard quantization methods borrowed from quantum field theory to more sophisticated points of view upon spacetime characterized by efforts to rethink the very structure of spacetime in terms of different mathematical objects other than ordinary points. These unconventional approaches are motivated basically by the problem of divergences in quantum gravity which arises when one follows the standard methods[1,2].

Very recently a very attractive description of quantum gravity, rather in the framework of standard methods, was given by Landi and Rovelli[3]. Their results are based on some previous works by Chamseddine and Connes done in the framework of noncommutative geometry[4]. Connes showed that there is a relationship between the geometry of a Riemannian spin manifold and the algebra generated by the Dirac operator together with smooth functions on spacetime. Moreover, once the later is known, the former can be recovered and the action of general relativity can be given algebraically as the Dixmier trace of some function of the Dirac operator[4–8]. These results express the fact that the Dirac operator can be used instead of the metric to describe the geometry of spacetime. Then its eigenvalues, which are diffeomorphism invariant objects, can be taken as dynamical variables of the gravitational field, which is exactly what Landi and Rovelli did. They showed that Poisson brackets can be expressed in terms of energy-momentum tensor of eigenspinors which is the Jacobian matrix of transformation from the metric to eigenvalues and that Einstein equations can be derived from a spectral action with no cosmological term. These very interesting results are plagued somehow by the applicability of noncommutative geometry to only Euclidean case. Indeed, when one tries to extend the noncommutative geometry to spacetime, one faces an obvious obstruction that comes from the fact that spacetime, at macroscopic scale, has a Lorentzian structure while the noncommutative geometry encodes the geometry of a Riemannian spin manifold into a real spectral triple[4]. The difference between Lorentzian and Riemannian is precisely the obstruction here because in the Lorentzian case one cannot give a natural positive definite inner product for spinors on spacetime.If the positiveness is sacrified, then the Dirac operator is no longer self-adjoint and thus the real spectral triple is no longer defined. Moreover, only on a Riemannian space-time manifold, the Dirac operator is elliptic. However, there are tentativs to find out ways around this problem and an interesting geometric construction based on a foliation of spacetime into space-like hypersurfaces can be found in[9]. Nevertheless, the Euclidean case is quite interesting by itself to merit further study.

It is the aim of this paper to investigate whether the eigenvalues of the Dirac operator can be used to describe Euclidean supergravity. This kind of system has been extensively studied lately mainly in the framework of path integral quantization of supergravity with a stress on the problem of the boundary conditions which are to be imposed on the fermions[10,11]. As we shall see, the extension to the minimal supergravity is possible, but there are several constraints that must be imposed on the gravity supermultiplet as well as on the eigenspinors of the Dirac operator. The origin of these constraints roots in the requirement that eigenvalues be gauge invariant functions or dynamical variables of the system. If we require further that the primary constraints be the same after covariant transport along two different paths

between two points, we obtain some restrictions on the possible spacetime manifolds, more precisely on the curvatures of tangent bundle and spinor bundle, respectively. The form of the corresponding equations is deduced under some siplifying assumptions and it is shown that manifolds for which both curvatures vanish satisfy these equations. For a more general discussion the reader is referred to a forthcoming paper[19] since the discussion in that case is too extensive and presents its own distinctive problems to be included in the present study.

The outline of the present paper is as follows. In Sec.II we review the main results obtained in the case of general relativity. In Sec.III we present the construction of covariant phase space and briefly discuss the Dirac operator when local supersymmetry is considered. The relations that must be satisfied by supermultiplets as well as the constraints on eigenspinors are derived in Sec.IV. In Sec. V we determine the form of the restrictions that a spacetime manifold should obey in order that the primary constraints mantain their form after a covariant transport. In Sec.VI we discuss several aspects of the theory and make some concluding remarks. The Appendix A reviews some definitions from the theory of the elliptic operators necessary in the discussion of the Dirac operator. The Appendix B presents the action of the two covariant derivatives used in this paper while the Appendix C shows the most important relations necessary to deduce the relations in Sec.V. We use units such that $8\pi G = 1$.

II. GENERAL RELATIVITY IN TERMS OF DIRAC EIGENVALUES

To make this paper relatively self-contained we will review the results obtained in the case of gravity described by Dirac eigenvalues[3]. We work on a compact 4D (spin) manifold without boundary M and we formulate general relativity in terms of tetrad fields $e^a_\mu(x)$, where $\mu = 1, \cdots, 4$ are spacetime indices and $a = 1, \cdots, 4$ are internal Euclidean indices raised and lowered by the Euclidean metric δ_{ab}. The metric field is $g_{\mu\nu}(x) = e^a_\mu(x) e_{\nu a}(x)$ and the spin connection $\overset{\circ}{\omega}{}^a_{\mu b}$ is defined by $\partial_{[\mu} e^a_{\nu]} = \overset{\circ}{\omega}{}^a_{\mu b}\, e^b_\nu$. The phase space of the system is covariant and is defined as the space of all solutions of the equations of motion, modulo gauge transformations. In this case the gauge transformations are composed by 4D diffeomorphisms and local rotations of the tetrad fields and thus the phase space contains equivalence classes of tetrad fields. At the same time, the phase space can be identified with the space of the gauge orbits on the constraint surface and with the space of Ricci flat 4-geometries. Let us denote the space of the smooth tetrad fields by $\mathcal{E}$ and the space of orbits of the gauge transformations in $\mathcal{E}$ by $\mathcal{G}$. The functions on the phase space are called *observables* and, technically speaking, they are functions on the constraint surface that commute with all the constraints.

On the manifold M there is a natural elliptic operator, namely the Dirac operator $\overset{\circ}{D}$. The ellipticity of $\overset{\circ}{D}$ means that the symbol of $\overset{\circ}{D}$ denoted by $\sigma_v(\overset{\circ}{D})$ is an isomorphism (see the Appendix A). In terms of tetrads and components of spin connection the Dirac operator is given by

$$\overset{\circ}{D} = i\gamma^a e^\mu_a (\partial_\mu + \overset{\circ}{\omega}_{\mu bc} (e, \psi) \sigma^{bc}), \tag{1}$$

where

$$\overset{\circ}{\omega}_{\mu bc} = \frac{1}{2} e^\nu_b (\partial_\mu e_{c\nu} - \partial_\nu e_{c\mu}) + \frac{1}{2} e^\rho_b e^\sigma_c \partial_\sigma e_{\rho d} e^d_\mu - (b \leftrightarrow c) \tag{2}$$

and γ^a's form an Euclidean representation of the Clifford algebra C_4, i. e. $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$. We can see from Eq.(1) that the Dirac operator is naturally defined for each geometrical structure on M. Furthermore, for each set of tetrad fields, $\overset{\circ}{D}$ is self-adjoint on the Hilbert space of spinor fields with a scalar product

$$< \psi, \phi >= \int d^4x\sqrt{g}\psi^*(x)\phi(x) \tag{3}$$

where ψ^* represents the complex conjugate of ψ. Since M is a compact manifold, $\overset{\circ}{D}$ admits a discrete spectrum of real eigenvalues and a complete set of eigenspinors

$$\overset{\circ}{D}\overset{\circ}{\chi}^n = \overset{\circ}{\lambda}^n \overset{\circ}{\chi}^n \tag{4}$$

where $n = 0, 1, 2, \cdots$. Because $\overset{\circ}{D}$ depends on e , $\overset{\circ}{\lambda}^n$'s define a discrete family of real valued functions on $\mathcal{E}$ and a function from $\mathcal{E}$ into the space of infinite sequences R^∞

$$\overset{\circ}{\lambda}^n \ : \mathcal{E} \longrightarrow R \quad , \quad e \rightarrow \overset{\circ}{\lambda}^n (e) \tag{5}$$

$$\overset{\circ}{\lambda}^n \ : \mathcal{E} \longrightarrow R^\infty \quad , \quad e \rightarrow \{\overset{\circ}{\lambda}^n (e)\}. \tag{6}$$

The point here is the fact that, for every n, $\overset{\circ}{\lambda}^n$ is invariant under diffeomorphisms of M as well as under rotations of tetrad fields. Therefore they define a set of observables of general relativity. It is worthwhile to notice that it is possible that $\overset{\circ}{\lambda}^n$'s do not coordinate neither the space of gauge orbits nor the phase space. That happens any time when gauge equivalent tetrad fields have different spectra. Then, because it is possible to find two metric fields with the same spectra, $\overset{\circ}{\lambda}^n$'s do not define an injective function.

The above construction allows us to define a Poisson structure on the set of eigenvalues. That is possible since there is a symplectic two-form Ω on the phase space given by

$$\Omega(X, Y) = \frac{1}{4}\int_\Sigma d^3\sigma n_\rho [X^a_\mu, \overset{\leftrightarrow}{\nabla}_\tau Y^b_\nu]\epsilon^\tau_{abv}\epsilon^{v\rho\mu\nu}, \tag{7}$$

where $X^a_\mu[e]$ define a vector field on the phase space and the brackets are given by

$$[X^a_\mu, \overset{\leftrightarrow}{\nabla}_\tau Y^b_\nu] = X^a_\mu \nabla_\tau Y^b_\nu - Y^a_\mu \nabla_\tau X^b_\nu. \tag{8}$$

Here Σ is an arbitrary Arnowitt-Deser-Misner surface an n_ρ is its normal one form. Using the inverse of the symplectic form matrix we can write down the Poisson bracket of any two eigenvalues

$$\{\overset{\circ}{\lambda}^n, \overset{\circ}{\lambda}^m\} = 4\int d^4x \int d^4y T^{[n\mu}_a(x) P^{ab}_{\mu\nu}(x, y) T^{m]\nu}_b(y), \tag{9}$$

where $P^{ab}_{\mu\nu}$ is the inverse of the symplectic form matrix and $T^{m\nu}_b(x)$ is the energy-momentum tensor of the spinor $\overset{\circ}{\chi}^n$ in tetrad notation and it represents the Jacobian matrix of the transformation from e to $\overset{\circ}{\lambda}^n$[3].

As shown in[5,6], the gravitational action in units $\hbar = c = G = 1$ can be written as the Dixmier trace of a simple function on the Dirac operator

$$S = Tr[\gamma(\overset{\circ}{D})] \tag{10}$$

where γ is a smooth monotonic function of the Dirac operator such that

$$\gamma(x) = \begin{cases} 1 \text{if } x < 1 - \delta \\ 0 \text{if } x > 1 + \delta \end{cases} \tag{11}$$

where $\delta << 1$. Then S represents the number of eigenvalues of the Dirac operator smaller than 1 once the gravitino is fixed. One can also write the action in terms of eigenvalues

$$S_1[\overset{\circ}{\lambda}] = \sum \gamma_1(\overset{\circ}{\lambda}), \tag{12}$$

where $\gamma_1(x) = \gamma(x) - \epsilon^4 \gamma(\epsilon x)$, $\epsilon << 1$. Moreover, the Dirac eigenvalues are not all independent and thus they cannot be simply varied in S_1 .

There are some other interesting conclusions that can be drawn from this variant of quantum gravity. However, because they would stray us away from the subject, the reader is reffered to[3] for other interesting details.

III. SUPERGRAVITY IN TERMS OF DIRAC EIGENSPINORS

To extend the ideas presented in the previous section we have to repeat firstly the same geometrical construction in the supersymmetric case. If we consider a supersymmetric partner of the graviton and we impose the local supersymmetry transformations then we get Euclidean supergravity.

Consider Euclidean minimal supergravity on M. The graviton is represented in the tetrad formalism by the fields e^a_μ. To have a local supersymmetry we must assign to the graviton a gravitino which must be a Majorana spinor. There is a problem here because, as is known, the group $SO(4)$ which is the local rotation group of tetrads in the Euclidean case, admits no Majorana spinor representation. Indeed, we can find no $SO(4)$ spinor that can satisfy Majorana condition $\psi^\dagger \gamma_4 = \psi^T C$. Fortunately, there is a standard way of makeshifting around the problem. It is known that in the Euclidean case the following relation can be written: $\bar{\psi} = \psi^T C$. Now if we define the adjoint spinors as the ones which satisfy the previous Majorana conjugation relation we obtain the desired Majorana spinors of the Euclidean theory. This convention does not affect the Lorentzian theory in which the fermions are Majorana spinors. The only difference appears in the Euclidean theory which makes now no reference to $\psi^\dagger$. With this definition of a Majorana spinor at hand the minimal gravity supermultiplet of the theory has the right number of degrees of freedom for both bosonic and fermionic partners[11,12]. We must say that, since we want to construct gauge invariant quantities, we are interested in solutions of the equations of motion. Therefore, it is enough to consider on-shell supersymmetry. In this case the supersymmetric algebra closes over graviton and gravitino only. Off-shell, the supersymmetry usually requires six more bosonic fields since there is a mismatch of the bosonic and fermionic degrees of freedom and the supermultiplet must be enlarged over these nonpropagating fields accordingly.

We define the phase space of Euclidean supergravity exactly as in general relativity, namely as the space of the solutions of the equations of motion modulo the gauge transformations[13]. The gauge transformations are 4D diffeomorphisms, local $SO(4)$ rotations and local $N = 1$ supersymmetry. The covariant phase space is then the space of all superpartners (e, ψ) that are solutions of the equations of motion modulo diffeomorphisms, internal rotations and local supersymmetry. As in the nonsupersymmetric case, the observables of the theory are functions on the phase space. Our main purpose is to see in what circumstances the eigenfunctions of the Dirac operator can define a set of observables of Euclidean supergravity. To this end we must analyze the Dirac operator in the presence of supersymmetry.

On a given spin manifold, the Dirac operator is the most fundamental differential operator. In even dimensions the spinor space divides in half depending on the eigenvalues of the chirality operator Γ_{D+1} defined as usual

$$\Gamma_{D+1} = \Gamma_1 \Gamma_2 \cdots \Gamma_D \tag{13}$$

where Γ's are Dirac matrices. The chiral operator can have two real eigenvalues and therefore any spinor enters one of the equivalence classes defined by these two eigenvalues on the space of spinors. We can write that as

$$\Gamma_{D+1} \psi_{\pm} = \pm \psi_{\pm} \tag{14}$$

where ± 1 are the two eigenvalues of the chiral operator. The Dirac operator is a first order operator acting between the two chiral bundles $C^{\infty}(S^{\pm})$. We mention that an elliptic complex can be obtained from it by tensoring with S^{-}, a construction that is well known in index theory[15]. As we mentioned in the previous section, the compactness of M ensures that the Dirac operator has a discrete spectrum and this spectrum depends on each geometrical configuration described by tetrad fields.

Now if we consider the supersymmetric case, the local supersymmetry requires the addition of an extra term to the Dirac operator which is of the form

$$D = \overset{\circ}{D} + K \tag{15}$$

where $\overset{\circ}{D}$ is given by (1) and K is given by

$$K = i \gamma^a e^{\mu}_a K_{\mu bc}(\psi) \sigma^{bc} \tag{16}$$

where

$$K_{\mu ab}(\psi) = \frac{i}{4} (\bar{\psi_\mu} \gamma_a \psi_b - \bar{\psi_\mu} \gamma_b \psi_a + \bar{\psi_b} \gamma_\mu \psi_a), \tag{17}$$

where $\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$ and $\overset{\circ}{D}$ is defined with σ^{ab}, too. If we consider that the Dirac operator is defined on the full spin bundle SM, or more precisely, on the sections of it $\Gamma(SM)$, then D is an elliptic operator. Indeed, $\overset{\circ}{D}$ depends only on the graviton , i. e. on $(e, 0)$ from the supermultiplet and K depends only on the gravitino $(0, \psi)$, while D depends on (e, ψ) which is the full gravitational supermultiplet. Thus, the symbol $\sigma_v(D)$ of the Dirac operator in the presence of supersymmetry differs from $\sigma_v(\overset{\circ}{D})$ by a map

$$L_0' : U \to Hom(SM, SM) \quad , \quad U \subset M. \tag{18}$$

(For the definition of the symbol see Appendix A. The reader might like to consult also[16]). It is this map that is assigned to the term that depends on the gravitino in (15). On the full spin bundle $\sigma_v(\overset{\circ}{D})$ is an isomorphism and $K(\psi)$, once the gravitino fixed, raises an isomorphism, too. That implies that L_0' added to the symbol of the Dirac operator in the nonsupersymmetric case does not affect its property of being an isomorphism. Therefore, $\sigma_v(D)$ is an isomorphism at its turn and from here results that D is an elliptic differential operator on M in the presence of local supersymmetry. Now if M is compact as we have already assumed, D has a discrete spectrum and a complete set of eigenspinors so that we can write

$$D\chi^n = \lambda^n \chi^n \tag{19}$$

where $n = 0, 1, 2, \cdots$. Let us denote the space of all gravitational supermultiplets by $\mathcal{F}$. Then λ^n's define a discrete family of functions on $\mathcal{F}$ since these functions depend on (e, ψ) which is a consequence of the dependence of D on (e, ψ). Similar relations to (6), can be written down in the supersymmetric case

$$\lambda^n : \mathcal{F} \longrightarrow R \quad , \quad (e, \psi) \to \lambda^n(e, \psi) \tag{20}$$

$$\lambda^n : \mathcal{F} \longrightarrow R^\infty \quad , \quad (e, \psi) \to \{\lambda^n(e, \psi)\}. \tag{21}$$

In general the eigenvalues λ^n's are not invariant under the gauge transformations of Euclidean supergravity. Therefore we cannot immediately use λ^n's as observables. To do that we must see under what circumstances they are gauge invariant. It is clear now that by imposing the gauge invariance upon λ^n's we must obtain some constraints on the system. It is our purpose to find what are these constraints. They were reported for the first time in[18].

IV. CONSTRAINTS FROM GAUGE INVARIANCE OF DIRAC EIGENVALUES IN EUCLIDEAN SUPERGRAVITY

To derive the set of all possible constraints which make the eigenvalues of the Dirac operator gauge invariant we must impose the invariance of λ^n's with respect to each type of gauge transformation. Moreover, under the gauge transformations Eq.(19) also transforms. But if we require that λ^n's be dynamical variables, in the right-hand side of (19) there must appear the variation of eigenvalues which vanishes. The resulting equation is an equation on the eigenspinors of the Dirac operator. Therefore, from the gauge invariance of λ^n's we must obtain constraints on the set of eigenspinors of the Dirac operator.

Let us begin with a general variation of any eigenvalue under an infinitesimal gauge transformation. This is given by

$$\delta\lambda^n(e, \psi) = \frac{\delta\lambda^n}{\delta e^a_\mu}\delta e^a_\mu + \frac{\delta\lambda^n}{\delta\psi^\alpha_\mu}\delta\psi^\alpha_\mu = 0. \tag{22}$$

This is the fundamental relation which defines the gauge invariance of λ^n. To find all of the possible constraints we must put for the variations of the graviton and the gravitiono in (22) their corresponding infinitesimal variations under different types of gauge transformations.

The first type of gauge transformation is 4D diffeomorphism. This is generated by an infinitesimal vector field on M $\xi = \xi^\mu \partial_\mu$ where ξ^μ are infinitesimal. The variation of λ^n under it is given by the Lie derivatives acting on graviton and gravitino

$$\delta e^a_\mu = \xi^\nu \partial_\nu e^a_\mu \quad , \quad \delta\psi_\mu = \xi^\nu \partial_\nu \psi_\mu \tag{23}$$

because both e^a_μ and ψ_μ are vectors with respect to the index μ. The derivative of λ^n with respect to the graviton which enters the first term in (22) can be computed by deriving the scalar product $< \chi^n, D\chi^n >$ with respect to e^a_μ. If we consider that all eigenvectors are normalized we obtain

$$\frac{\delta\lambda^n}{\delta e^a_\mu} =< \chi^n | \frac{\delta}{\delta e^a_\mu} D | \chi^n >= \mathcal{T}^{n\mu}_a(x). \tag{24}$$

In deriving (24) we took into account the fact that $K_{\mu ab}$ does not depend on the graviton.

Here $\mathcal{T}^{n\mu}_a = T^{n\mu}_a + K^{n\mu}_a$ where $T^{n\mu}_a$ is the energy-momentum tensor of the spinor χ^n[3] and $K^{n\mu}_a =< \chi^n | i\gamma_a K^\mu_{bc}(\psi)\sigma^{bc} | \chi^n >$. In a similar manner one can evaluate the derivative of λ^n with respect to the gravitino field. This is given by the derivative of the same scalar product as in (24) with respect to the gravitino. The only term contributing to this derivative is K and we obtain after simple calculations

$$\frac{\delta\lambda^n}{\delta\psi^\alpha_\mu} = \frac{i}{4}\int \sqrt{e}\chi^{n*}\gamma^a e^\nu_a [\bar{\psi}^\beta_\nu (\gamma_b)_{\alpha\beta} e^\mu_c - \bar{\psi}^\beta_\nu (\gamma_c)_{\alpha\beta} e^\mu_b + \bar{\psi}^\beta_b (\gamma_\nu)_{\alpha\beta} e^\mu_c]\sigma^{bc}\chi^n = \Gamma^{n\mu}_\alpha. \tag{25}$$

Eq.(25) represents nothing else but the matrix elements of K on the eigenstates of D. If we put together (25), (23) and (22) and if we consider that the variation of λ^n must vanish for arbitrary ϵ^ν we obtain the following equation:

$$\mathcal{T}^{n\mu}_a \partial_\nu e^a_\mu - \Gamma^{n\mu}_\alpha \partial_\nu \psi^\alpha_\mu = 0. \tag{26}$$

This is a first set of constraints that must be imposed on the supermultiplet (e, ψ). In a similar manner the invariance of the eigenvalues under $SO(4)$ leads to new constraints. The fields e and ψ transform under an infinitesimal $SO(4)$ rotation as a vector and a spinor, respectively. The corresponding relations are

$$\delta e^a_\mu = \theta^{ab} e_{b\mu} \tag{27}$$

$$\delta\psi^\alpha_\mu = \theta^{ab}(\sigma_{ab})^\alpha_\beta \psi^\beta_\mu \tag{28}$$

where $\theta_{ab} = -\theta_{ba}$ parametrize an infinitesimal rotation and $\sigma^{ab} = i\Sigma^{ab}$. The infinitesimal transformation of an eigenvalue is given by the basic relation (22) where now (28) must be taken into account for the infinitesimal variation of the supermultiplet while the derivatives of λ^n with respect to e and ψ remain the same as above. Then the following equation is straightforward

$$\mathcal{T}^{n\mu}_a e_{b\mu} + \Gamma^{n\mu}\sigma_{ab}\psi_\mu = 0. \tag{29}$$

Eq.(29) form the second set of constraints that must be imposed on the phase space of the theory. It is equally easy to derive the last set of constraints on (e, ψ). They come

from the invariance of the Dirac eigenvalues under the local $N = 1$ supersymmetry. Under an infinitesimal on-shell supersymmetry transformation the supermultiplet transforms as follows

$$\delta e^a_\mu = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu \qquad \delta\psi_\mu = \mathcal{D}_\mu\epsilon \tag{30}$$

where $\epsilon(x)$ is an infinitesimal Majorana spinor field, i. e. it obeys the Majorana conjugation relation $\bar{\epsilon} = \epsilon^T C$. Here $\mathcal{D}_\mu$ is the non-minimal covariant derivative acting on spinors. There is another minimal covariant derivative which acts on tensors and which is expressed in terms of Christoffel symbols (see Appendix B). Under (30) the spin connection transforms as

$$\delta\omega^{ab}_\mu = A^{ab}_\mu - \frac{1}{2}e^b_\mu A^{ac}_c + \frac{1}{2}e^a_\mu A^{bc}_c \tag{31}$$

where

$$A^{\mu\nu}_a = \bar{\epsilon}\gamma_5\gamma_a\mathcal{D}_\lambda\psi_\rho\epsilon^{\nu\mu\lambda\rho}. \tag{32}$$

Now to derive the constraints imposed by the local supersymmetry we start as in the diffeomorphism case and the rotation cases from the (22) and we take for the variations of e and ψ (30). Then we obtain the following equation:

$$\mathcal{T}^{n\mu}_a\bar{\epsilon}\gamma^a\psi_\mu + \Gamma^{n\mu}\mathcal{D}_\mu\epsilon = 0. \tag{33}$$

which is the expression of the constraints that must be imposed on the phase space if λ^n are invariant under $N = 1$ local supersymmetry.

Let us examine now what are the consequences of the gauge invariance of Dirac eigenvalues upon the Dirac eigenspinors. If we start with the eigenvalue problem (19) and transform it under an infinitesimal gauge transformation its variation reads

$$\delta D\chi^n = (\lambda^n - D)\delta\chi^n \tag{34}$$

where we considered that λ^n's are invariant under an infinitesimal gauge transformation. If the gauge transformation is a diffeomorphism of M generated by an infinitesimal vector field $\epsilon = \epsilon^\nu\partial_\nu$ we have the variations in (34)

$$\delta D = \pounds_\epsilon D = [\epsilon, D] \quad , \quad \delta\chi^n = \pounds_\epsilon\chi^n = \epsilon^\nu\partial_\nu\chi^n. \tag{35}$$

We use in (35) the expression of D given by (15). Then a short and simple algebraic calculus gives us the variation of D. Using some short-hand notations for the terms entering this variation we obtain

$$\delta D = i\gamma^a[b^\mu_a(\epsilon)\partial_\mu + f_a(\epsilon)] = [b^\mu(\epsilon)\partial_\mu + f(\epsilon)] \tag{36}$$

where we have used the following notations:

$$b^\mu(\xi) = i\gamma^a b^\mu_a(\xi) \quad , \quad b^\mu_a(\xi) = \xi^\nu\partial_\nu e^\mu_a - e^\nu_a\partial_\nu\xi^\mu - 2e^\nu_a\xi^\mu\omega_{\nu bc}\sigma^{bc} \tag{37}$$

$$f(\xi) = i\gamma^a\xi^\nu\partial_\nu(e^\mu_a\omega_{\mu bc})\sigma^{bc}. \tag{38}$$

Then from (35) and (36) we obtain the following equation:

$$\{[b^\mu(\xi) - c(\lambda,\xi)^\mu]\partial_\mu + f(\xi)\}\chi^n = 0 \tag{39}$$

where

$$c(\lambda,\xi)^\mu = (\lambda^n - D)\xi^\mu \tag{40}$$

arises from the variation of χ^n. Eq.(39) represents a first set of equations that must be satisfied by the eigenspinors of the Dirac operator as a consequence of the invariance of eigenvalues under diffeomorphisms. In the case of an $SO(4)$ rotation the transformation of D is slightly more complicated since now $\omega_{\mu ab}$ transform as gauge fields. In this case e_a^μ transforms as a vector and χ^n transforms as a spinor. The transformation of $\omega_{\mu ab}$ under rotations is given by

$$\delta\omega_{\mu ab} = i[\theta\sigma, \omega_{\mu ab}] - i\partial_\mu\theta\sigma M_{ab} \tag{41}$$

where $\theta_{ab} = -\theta_{ba}$ parametrize an infinitesimal $SO(4)$ rotation and $\theta\sigma = \theta_{ab}\sigma_{ab}$. The variations of D and χ^n can be obtained after some simple algebra and they are

$$\delta D = \theta_a^a D - \gamma^a e_a^\mu\{[\theta\sigma, \omega_{\mu cd}] - \partial_\mu\theta\sigma M_{cd}\}\sigma^{cd} \tag{42}$$

$$\delta\chi^{n\alpha} = i(\theta_{ab}\sigma_{ab})^\alpha_\beta\chi^{n\beta}. \tag{43}$$

Now if we introduce the following notations:

$$g(\theta) = [\gamma^c e_c^\mu([\theta\sigma, \omega_{\mu ab}] - \partial_\mu\theta\sigma M_{ab})]\sigma^{ab} \tag{44}$$

$$h(\theta) = i(\lambda^n - D)\theta\sigma \tag{45}$$

and introduce (43) in (34) the constraints coming from $SO(4)$ invariance can be written as

$$[\theta_a^a D - g(\theta) + h(\theta)]\chi^n = 0. \tag{46}$$

Finally, if $N = 1$ local supersymmetry is considered, the left-hand side of (34) vanishes since χ^n are inert under this symmetry. Thus, our equation becomes $\delta D\chi^n = 0$. In this case e_μ^a and ψ_μ tranform accordingly to (30). Now considering (30) and (31) we can easily write down the variation of D which leads us immediately to the supersymmetric constraint. The calculus raising no problem and being quite simple we write down only the result in a notation designed to make it more transparent

$$[j_a^\mu(\epsilon)\partial_\mu + k_a(\epsilon) + l_a]\chi^n = 0 \tag{47}$$

where

$$j_a^\mu(\epsilon) = \frac{1}{2}\gamma_a\bar{\epsilon}\psi^\mu \ , \quad k_a(\epsilon) = \frac{1}{2}\gamma_a\bar{\epsilon}\psi^\mu\omega_{\mu cd}\sigma^{cd} \tag{48}$$

$$l_a = e_a^\mu[B_{\mu cd} - \frac{1}{2}e_{\mu d}B_{ec}^e + \frac{1}{2}e_{\mu c}B_{ed}^e]\sigma^{cd}. \tag{49}$$

and $\bar{\epsilon}$ is an infinitesimal Majorana spinor. The equations (26), (29) and (33) represent constraints on the phase space which must be imposed in order to have Dirac eigenvalues as

observables of the theory. In this respect, they are primary constraints, i.e. they come first into discussion when the dynamical variables are discussed. As consequences of these primaries follow the equations (39), (46) and (47) which restrict the set of eigenspinors to those which satisfy these relations. It is fair to say that all these equations are highly nontrivial. A method to solve them is unknown to the author at present. It would be interesting to see whether the constraint equations have any solution because there is no obvious evidence that this is the case. For example, even if the conditions of gauge invariance of Dirac eigenvalues are fulfilled, it might be possible that there be no eigenspinor to satisfy the constraints on spinors. That, in turn, would mean that there are no eigenspinors compatible to this description of Euclidean supergravity which would question the validity of this approach. The lesson to be learnt from here is that both sets of equations must have nontrivial solutions in order to have a consistent formulation of Euclidean supergravity in terms of Dirac eigenspinors.

Let us point out some subtleties which would appear in a possible quantization of Euclidean supergravity in terms of Dirac eigenvalues. As in the gravity case, the information about the geometry of the manifold in the presence of local supersymmetry is encoded in λ^n's. At a first glance, in the process of quantization the eigenspinors could be left aside since they do not appear directly and we should pay attention only to the eigenvalues. This conclusion is not true. As we saw in Sec. II, the Poisson brackets of λ^n's are determined by energy-momentum tensor of the eigenspinors and thus the eigenspinors become important in the quantization process. In the supersymmetric case the situation is somehow similar, but the fact that the system is subjected to several constraints makes it more suitable for BRST quantization[14]. The eigenspinors intervene in quantization precisely through these constraints. To see how this comes, let us denote the constraints (26), (29) and (33) with $\Sigma^n(\mathcal{T},\Gamma)=0$, $\Theta^n_{ab}(\mathcal{T},\Gamma)=0$ and $\Phi^n(\mathcal{T},\Gamma)=0$. Here $\mathcal{T}$ and Γ denote $T^{n\mu}_a$ and $\Gamma^{n\mu}_a$, respectively. The path integral can be written using the Fadeev-Popov trick as

$$\mathcal{Z}\sim\int D[e]D[\psi]e^{-S_0}\sim\int D[e]D[\psi]D[\sigma]D[\tau]D[\phi]e^{-S} \tag{50}$$

where the first integral is factorized with the volume of all of the gauge transformations and

$$S = S_0+S_1+S_2 \tag{51}$$

$$S_1 = \int(\sigma_n\Sigma^n(\mathcal{T},\Gamma)+\tau^{ab}_n\Theta^n_{ab}(\mathcal{T},\Gamma)+\phi_n\Phi^n(\mathcal{T},\Gamma)) \tag{52}$$

$$S_2 = \int(s_n\delta_\alpha\Sigma^n(\mathcal{T},\Gamma)+t^{ab}_n\delta_\alpha\Theta^n_{ab}(\mathcal{T},\Gamma)+f_n\delta_\alpha\Phi^n(\mathcal{T},\Gamma))c^\alpha \tag{53}$$

where σ_n,τ^{ab}_n and ϕ_n are the antighosts associated to the gauge averaging conditions, s_n,t^{ab}_n and f_n are the corresponding ghosts and c^α are the ghosts associated to the gauge transformations and denoted generically by δ_α. We observe that the quantities $\mathcal{T}^{n\mu}_a$ and $\Gamma^{n\mu}_\alpha$ enter the path integral. But they are computed as matrix elements between eigenspinors of the Dirac operator and these eigenspinors must be solutions of (39), (46) and (47). This implies that one should take only those matrix elements of $\mathcal{T}$ and Γ that are obtained in those eigenstates of the Dirac operator which also solve the above constraints. Obviously, this leads to a certain simplification of the path integral. But the price to be paid for this is solving the constraints on the eigenspinors, which in turn implies solving the primary constraints. To illustrate this point we could have employed a more complete formulation of the BRST

theory, like BV or BFV, but the problems remain the same. Since our discussion has the role to emphase the difficulties which arise in the quantization of Euclidean supergravity, we paid no attention to the structure of gauge transformation which is vital for the quantum theory. However, as long as the constraints of the theory are not solved, any further discussion of the quantization has just a general character. The matter deserves a deeper study, but that is out of the line of the present paper.

V. GLOBAL CONSISTENCY OF PRIMARY CONSTRAINTS

Primary constraints (26), (29) and (33) have a rather local character. However, to obtain a theory fully consistent, it would be desirable that these constraints be compatible with the global structure of M. As usual, when a global problem is addressed, this compatibility might restrict the possible manifolds that can support the theory. If secondary constraints are promoted to global constraints, too, further restrictions may appear. However, since secondaries arise as consequences of primaries and since they essentially restrict only the spinors belonging to the Dirac eigenspinors set, we do not investigate this matter here.

The main idea which will be used in what follows to study the consequences of globality of primary constraints, is to transport these equations from one point of M to another one along two different paths. If we require that the same result be obtained after the two transports we find some restrictions upon the structure of M.

The transport of any of the constraints will be implemented via the exponentiation of covariant derivative, similar to the exponentiation of Lie derivative[1]. Since the constraints are made out of objects which are composed by bosons and fermions, such as $\mathcal{T}_a^{n\mu}$ or $\Gamma_\alpha^{n\mu}$, we must consider a covariant derivative acting on bosons and another one acting on fermions. These are associated to two connections ∇ and ∇^S on the tangent bundle TM and spinor bundle SM respectively

$$\begin{aligned} \nabla &: \mathcal{X}(M) \to Hom_R(TM, TM) \\ \nabla^S &: \mathcal{X}(M) \to Hom_R(SM, SM), \end{aligned} \tag{54}$$

where $\mathcal{X}(M)$ represents the algebra of vector fields on M. Due to the composite structure of constraints, the natural way to transport them from one point to another is to transport firstly all of the elementary objects as graviton, gravitino and the derivative, and then to reconstruct more complicated objects as the components of spin connection or of K term and then to write down the whole equation.

Let us assume that we have two congruences $c(\lambda)$ and $d(\mu)$ on M, where λ and μ are the parameters of the curves, and let us select a curvilinear rectangle at the intersection of the two congruences

$$\{Q, P, R, S, \} \in c(\lambda) \cap d(\mu) \tag{55}$$

---

[1]For spinors there is difficult to define Lie derivative along an arbitrary vector field, but if this vector field is chosen to have a particular form, for example to be a conformal Killing vector, the problem is removed.

where $|QP| \in c$, $|RS| \in c$, $|PR| \in d$, $|QS| \in d$ and in this case $|\ \ |$ denotes the curvilinear segment. Let us assume that the lengths of the sides of the rectangle are λ and μ measured in units of natural parameters of $c(\lambda)$ and $d(\mu)$, respectively. Suppose further that there are two vector fields ξ and η from $\mathcal{X}(M)$ such that ξ is defined along $c(\lambda)$ and η is defined along $d(\mu)$ and

$$[\xi, \eta] = 0. \tag{56}$$

As a particular case we can take $\xi = d/d\lambda$ and $\eta = d/d\mu$.

Now any object A can be transported, say, from Q to P along $c(\lambda)$ and the result will be

$$A(P) = e^{\lambda \bar{\nabla}_\xi} A(Q) \tag{57}$$

where $\bar{\nabla}$ stands for either ∇ or ∇^S and

$$\bar{\nabla}_\xi A(Q) = \xi^\nu \mathcal{D}_\nu A(Q) \tag{58}$$

where $\mathcal{D}_\nu$ is the covariant derivative. It acts on bosons minimally and on fermions non-minimally acccording to the requirements of supersymmetry (see Appendix B). The minimal covariant derivative is related to ∇ and the nonminimal one to ∇^S.

In what follows we are interested in transporting the constraints along $Q \to P \to R$ which we call path 1 and along $Q \to S \to R$ which we call path 2. Any object carrying the subscript 1 or 2 will be understood as transported along the respective path. Thus, for example, for a boson transported along path 1 we have

$$A_1 = e^{\mu \nabla_\eta} e^{\lambda \nabla_\xi} A. \tag{59}$$

An equality between two objects transported along path 1 or path 2 should hold at all orders in power expansion of exponentials in (59). We expect that the coefficients of $\mu\lambda$ capture some information about the structure of M. To make this term important we take μ and λ be conveniently small and thus the series in (59) truncate to

$$A_1 = A_1^{(0)} + \mu A_1^{(1)} + \lambda A_1^{(2)} + \frac{\mu^2}{2} A_1^{(3)} + \frac{\lambda^2}{2} A_1^{(4)} + \mu\lambda A^{(5)}. \tag{60}$$

Before moving to the transport of constraints, let us make some general remarks that will ease the forthcoming calculus otherwise pretty heavy and long. Any of the equations (26), (29) and (33) consists of a sum of two terms each of these being written as a product AB. Now when we compare the transported equations along the two paths we come to sums of two terms of the form $A_1 B_1 - A_2 B_2$ where A_1 and B_1 can be cast into the form (60). If we remark that under the change of paths 1$\leftrightarrow$2 (60) displays the following symmetry:

$$\begin{array}{ccc} A_1^{(0)} \leftrightarrow A_2^{(0)} & , & A_1^{(3)} \leftrightarrow A_2^{(4)} \\ A_1^{(1)} \leftrightarrow A_2^{(2)} & , & A_1^{(4)} \leftrightarrow A_2^{(3)} \\ A_1^{(2)} \leftrightarrow A_2^{(1)} & , & A_1^{(5)} \leftrightarrow \bar{A}_2^{(5)}, \end{array} \tag{61}$$

where the change of paths implies in fact

$$\mu \leftrightarrow \lambda \quad , \quad \eta \leftrightarrow \xi \tag{62}$$

and where we have considered that $\bar{A}^{(5)}$ is the same as $A_2^{(5)}$ but with the product $\nabla_\eta \nabla_\xi$ inverted. We must also notice that for any A, $A_1^{(0)} = A_2^{(0)} = A^0$. Taking (61) and (62) into account we see that

$$A_1 B_1 - A_2 B_2 = A^0 (B_{1NS}^{(5)} - B_{2NS}^{(5)}) + (A_{1NS}^{(5)} - A_{2NS}^{(5)}) B^0, \tag{63}$$

where the index NS indicates the non-symmetric part in ∇_η and ∇_ξ. The same holds true for fermions with the appropriate connection. The symmetry above can be easily demonstrated using the following relation:

$$e^{\mu \nabla_\eta} e^{\lambda \nabla_\xi} = 1 + \mu \nabla_\eta + \lambda \nabla_\xi + \frac{\mu^2}{2} \nabla_\eta \nabla_\eta + \frac{\lambda^2}{2} \nabla_\xi \nabla_\xi + \mu \lambda \nabla_\eta \nabla_\xi \tag{64}$$

which also holds for the spinorial case, with ∇ replaced by ∇^S.

Another important point that should be discussed concerns the Dirac eigenspinors χ^n. In general, an arbitrary spinor changes while it is transported along an arbitrary path. In the case of Dirac eigenspinors, this change is unwanted since it can take an eigenspinor χ^n out of the set of eigenspinors or it can move it onto another eigenspinor corresponding to a different eigenvalue χ^m. Both these changes alter our quantities $\mathcal{T}_a^{n\mu}$,$\Gamma_\alpha^{n\mu}$ and therefore we must require that they be zero. To this end, if we transport the Dirac equation along path 1 we must have

$$D_1 \chi_1^n = \lambda_1^n \chi_1^n \tag{65}$$

where D_1 is given by (15) with all of the entering objects transported along path 1. Spin connection is transported by transporting each graviton in it, according to (2). Analogously, for the K be transported we should firstly transport each fermionic component. The results are given in Appendix C. Using them we can write down the transported Dirac equation and we can power expand it. Acoording to the previous discusion we consider only the equivalent of (63) for this case. We are eventually led to

$$\begin{aligned}(D - \lambda^n)(R^S(\eta, \xi)\chi^n) = \{R(\eta, \xi)\lambda^n - i\gamma^a [(R(\eta, \xi) e_\mu^a)(\partial_\mu + \frac{1}{2} \overset{\circ}{\omega}_{\mu bc} \sigma^{bc}) \\ + i\gamma^a e_\mu^a (R(\eta, \xi)\partial_\mu + \frac{1}{2} \omega_{\mu bc}^{(5)} (R(\eta, \xi), R^S(\eta, \xi))\sigma^{bc})]\}\chi^n,\end{aligned} \tag{66}$$

where $R(\eta, \xi)$ and $R^S(\eta, \xi)$ are the curvatures of the tangent bundle TM and of the spinor bundle SM, respectively. $\omega_{\mu bc}^{(5)}(R(\eta, \xi), R^S(\eta, \xi))$ can be obtained from $\overset{\circ}{\omega}{}_{\mu bc}^{(5)}$ $(R(\eta, \xi))$ and $K_{\mu bc}^{(5)}(R^S(\eta, \xi))$ from Appendix C. The dependence of $\omega_{\mu bc}^{(5)}(R(\eta, \xi), R^S(\eta, \xi))$ on $R(\eta, \xi)$ and $R^S(\eta, \xi)$ means that in $\omega_{\mu bc}^{(5)}(R(\eta, \xi), R^S(\eta, \xi))$ the products $\nabla_\eta \nabla_\xi$ and $\nabla_\eta^S \nabla_\xi^S$ must be replaced by $R(\eta, \xi)$ and $R^S(\eta, \xi)$, respectively. Indeed, once that (56) holds true, the two curvatures are given by

$$R(\eta, \xi) = [\nabla_\eta, \nabla_\xi] \quad , \quad R^S(\eta, \xi) = [\nabla_\eta^S, \nabla_\xi^S]. \tag{67}$$

Equation (66) can be viewed as a restriction of possible curvatures of tangent bundle and spinor bundle once the eigenspinor χ^n is given. In principle, it should hold for all spectra of the Dirac operator. Thus we see that this equation imposes restrictions on the structure of M, namely on the two curvatures mentioned above. These restrictions are consequences of the promotion of the eigenspinors to global observables of Euclidean supergravity. The presence of the two curvatures in this equation is exactly the type of restriction we should have expected from this kind of treatment of the problem of global consistency of constraints. Notice that before discussing the transport of the Dirac equation along the two paths considered above we should have discussed a simple transport, i.e. the transport between just two arbitrary points. This would have led us to constraints on the two connections instead of constraints on the two curvatures. To see that, let us assume that we transport the Dirac equation only along $c(\lambda)$. The equation must hold true at the final point. Expanding the exponential in powers of λ we obtain an infinte set of equations. The first two of them are the following ones:

$$D\chi^n = \lambda^n \chi^n \tag{68}$$

at order zero and

$$[D(\nabla^S_\xi \chi^n) + i\gamma^a(\nabla_\xi e^\mu_a)(\partial_\mu + \frac{1}{2}\overset{\circ}{\omega}_{\mu bc}\,\sigma^{bc}) + i\gamma^a e^\mu_a((\nabla_\xi \partial_\mu) + \frac{1}{2}\omega^{\xi(0)}_{\mu bc}\sigma^{bc})]\chi^n = (\nabla_\xi \lambda^n)\chi^n + \lambda^n(\nabla^S_\xi \chi^n) \tag{69}$$

at first order. Here $\omega^{\xi(0)}_{\mu bc}$ can be obtained from $\omega^{(0)}_{\mu bc}$ if we set $exp(\mu\nabla_\eta)$ to one. The former equation is automatically satisfied while the second one remains as a constraint. The rest of constraints play an important or less important role depending on the magnitude of the parameter λ. Working in the real global case, that is when we move on an arbitrary distance on the curve $c(\lambda)$, the power expansion is less useful and we have to work with the exponentials instead. For a more detalied analysis of these issues we relegate the reader to the forthcoming paper[19].

An obvious simplification of (66), (68) and (69) can be obtained if the eigenspinors are subject to parallel transport along $c(\lambda)$ and $d(\mu)$

$$\nabla^S_\eta \chi^n = \nabla^S_\xi \chi^n = 0. \tag{70}$$

As a consequence, once χ^n is fixed in a point on M it remains the same as a spinor field. We shall assume in what follows that that is the case. A more general discussion can be found in[19].

Let us analyze what happens when the primary constraint (26) is transported along the two paths. To compute the changes in $\mathcal{T}^{n\mu}_a$, $\Gamma^{n\mu}_\alpha$, $\partial_\nu e^a_\mu$ and $\partial_\nu \psi^\alpha_\mu$ we use (64) and the Appendix C. After some tedious algebra the final results can be expanded in powers of λ and μ as in (60). As shown in (63), the essential terms are the coefficient of order zero and of $\mu\lambda$ non-symmetric in $\nabla_\eta\nabla_\xi$ and $\nabla^S_\eta\nabla^S_\xi$ (see Appendix C). Using them as well as (67) and (70) we obtain after some algebra the following relation:

$$< \chi^n | i\gamma^d \delta_{da}\delta^{\mu\rho}(\partial_\rho + \frac{1}{2}\overset{\circ}{\omega}{}^{(0)}_{\rho fg}\,\sigma^{fg}) -$$

$$\frac{1}{8}\gamma_a g^{\mu\nu}\sum_{(\nu,b,c)}[\bar{\psi}_\nu\gamma_b\psi_c]\sigma^{bc}|\chi^n>[(R(\eta,\xi))\partial_\nu)e^a_\mu+\partial_\nu(R(\eta,\xi)e^a_\mu)]+$$
$$<\chi^n|i\gamma^d\delta_{da}\delta^{\mu\nu}[(R(\eta,\xi)\partial_\nu)+\frac{1}{2}\overset{\circ}{\omega}{}^{(5)}_{\nu fgNS}(R(\eta,\xi))\sigma^{fg}]+$$
$$i\gamma^d[\frac{\delta}{\delta e^a_\mu}(R(\eta,\xi)e^\rho_d)(\partial_\rho+\frac{i}{2}\overset{\circ}{\omega}{}^{(0)}_{\rho fg}\sigma^{fg})-\frac{\delta}{\delta e^a_\mu}[(R(\eta,\xi)e_{\nu d}]\delta^{\nu\rho}(\partial_\rho+\frac{1}{2}\overset{\circ}{\omega}{}^{(0)}_{\rho fg}\sigma^{fg})]-$$
$$\frac{1}{8}\gamma_a g^{\mu\rho}\sum_{(\rho,b,c)}[\bar{\psi}_\rho\gamma_b(R^S(\eta,\xi)\psi_c)+(R^S(\eta,\xi)\bar{\psi}_\rho)\gamma_b\psi_c]\sigma^{bc}|\chi^n>\partial_\nu e^a_\mu+$$
$$\frac{1}{8}<\chi^n|\gamma^a e^\rho_a\frac{\delta}{\delta\psi^\alpha_\mu}[\sum_{(\rho,b,c)}[\bar{\psi}_\rho\gamma_b\psi_c]]\sigma^{bc}|\chi^n>[(R^S(\eta,\xi)\partial_\nu)\psi^\alpha_\mu+\partial_\nu(R^S(\eta,\xi)\psi^\alpha_\mu)]-$$
$$<\chi^n|i\gamma^a\{e^\rho_a\frac{\delta}{\delta\psi^\alpha_\mu}[\frac{i}{8}\sum_{(\rho,b,c)}[(R^S(\eta,\xi)\bar{\psi}_\rho)\gamma_b\psi_c+\bar{\psi}_\rho\gamma_b(R^S(\eta,\xi)\psi_c)]+[\delta^\mu_\nu\delta^\beta_\alpha(R(\eta,\xi)e^\rho_a)-$$
$$\frac{1}{8}\frac{\delta}{\delta\psi^\alpha_\mu}(R^S(\eta,\xi)\psi^\beta_\nu)e^\rho_a]\frac{\delta}{\delta\psi^\beta_\nu}[\sum_{(\rho,b,c)}[\bar{\psi}_\rho\gamma_b\psi_c]]\}\sigma^{bc}|\chi^n>\partial_\nu\psi^\alpha_\mu=0 \quad (71)$$

We can work out the second primary constraint (29) along the same line. Performing exactly the same steps we obtain the second restriction on the manifold M. Using again the results listed in the Appendix C we find the following equation:

$$<\chi^n|i\gamma^d\delta_{da}\delta^{\mu\nu}(\partial_\nu+\frac{1}{2}\overset{\circ}{\omega}{}^{(0)}_{\nu fg}\sigma^{fg})-$$
$$\frac{1}{8}\gamma_a g^{\mu\nu}\sum_{(\nu,d,c)}[\bar{\psi}_\nu\gamma_d\psi_c]\sigma^{dc}|\chi^n>(R(\eta,\xi)e_{b\mu})+<\chi^n|i\gamma^d\delta_{da}\delta^{\mu\nu}[(R(\eta,\xi)\partial_\nu)+$$
$$(72)$$
$$f12\,\overset{\circ}{\omega}{}^{(5)}_{\nu fgNS}(R(\eta,\xi))\sigma^{fg}]+i\gamma^d[\frac{\delta}{\delta e^a_\mu}(R(\eta,\xi)e^\rho_d)(\partial_\rho+\frac{1}{2}\overset{\circ}{\omega}{}^{(0)}_{\rho fg}\sigma^{fg})-$$
$$\frac{\delta}{\delta e^a_\mu}[(R(\eta,\xi)e_{\nu d}]\delta^{\nu\rho}(\partial_\rho+\frac{1}{2}\overset{\circ}{\omega}{}^{(0)}_{\rho fg}\sigma^{fg})]-$$
$$\frac{1}{8}\gamma_a g^{\mu\nu}\sum_{(\nu,d,c)}[\bar{\psi}_\nu\gamma_d(R^S(\eta,\xi)\psi_c)+(R^S(\eta,\xi)\bar{\psi}_\nu)\gamma_d\psi_c]\sigma^{dc}|\chi^n>e_{b\mu}+$$
$$<\chi^n|\gamma^d e^\rho_d\frac{\delta}{\delta\psi^\alpha_\mu}\frac{1}{2}[\sum_{(\rho,t,c)}[\bar{\psi}_\rho\gamma_t\psi_c]]\sigma^{tc}|\chi^n>[\sigma_{ab}]^\alpha_\gamma(R^S(\eta,\xi)\psi^\gamma_\mu)+$$
$$<\chi^n|i\gamma^d\{e^\rho_d\frac{\delta}{\delta\psi^\alpha_\mu}[\frac{1}{2}\sum_{(\rho,t,c)}[(R^S(\eta,\xi)\bar{\psi}_\rho)\gamma_t\psi_c+$$
$$\bar{\psi}_\rho\gamma_t(R^S(\eta,\xi)\psi_c)]+[\delta^\mu_\nu\delta^\beta_\alpha(R(\eta,\xi)e^\rho_d)+$$
$$\frac{1}{2}\frac{\delta}{\delta\psi^\alpha_\mu}(R^S(\eta,\xi)\psi^\beta_\nu)e^\rho_d]\frac{\delta}{\delta\psi^\beta_\nu}[\sum_{(\rho,t,c)}[\bar{\psi}_\rho\gamma_t\psi_c]]\}\sigma^{tc}|\chi^n>[\sigma_{ab}]^\alpha_\gamma\psi^\gamma_\mu=0 \quad (73)$$

In a similar manner we obtain the final restriction on the manifold from the third primary (33). The terms that multiply $\mathcal{T}^{n\mu}_a$ and $\Gamma^{\nu\mu}_\alpha$ are now $\bar{\epsilon}\gamma^a\psi_\mu$ and $\mathcal{D}_\mu\epsilon^\alpha$. The transport of the value of covariant derivative of ϵ^α from Q to R along of the path 1 is performed in a similar

manner to the transport of the Dirac operator which led us to (66). The calculations of the restriction imposed by the third primary constraint are somewhat more lenghty than the previous computations and heavily rely on the Appendix C, too. The final result is

$$
\begin{aligned}
&< \chi^n | i\gamma^d \delta_{da} \delta^{\mu\nu} (\partial_\nu + \frac{1}{2} \overset{\circ}{\omega}{}^{(0)}_{\nu fg} \sigma^{fg}) - \\
&\frac{1}{8}\gamma_a g^{\mu\nu} \sum_{(\nu,b,c)} [\bar{\psi}_\nu \gamma_b \psi_c] \sigma^{bc} | \chi^n > [(R^S(\eta,\xi)\bar{\epsilon})\gamma^a \psi_\mu + \bar{\epsilon}\gamma^a (R^S(\eta,\xi)\psi_\mu)] + \\
&< \chi^n | i\gamma^d \delta_{da} \delta^{\mu\nu} [(R(\eta,\xi)\partial_\nu) + \frac{1}{2} \overset{\circ}{\omega}{}^{(5)}_{\nu fgNS} (R(\eta,\xi))\sigma^{fg}] \\
&+ i\gamma^d [\frac{\delta}{\delta e^a_\mu} (R(\eta,\xi) e^\rho_d)(\partial_\rho + \frac{1}{2} \overset{\circ}{\omega}{}^{(0)}_{\rho fg} \sigma^{fg}) - \frac{\delta}{\delta e^a_\mu} [(R(\eta,\xi) e_{\nu d}] \delta^{\nu\rho} (\partial_\rho + \frac{1}{2} \overset{\circ}{\omega}{}^{(0)}_{\rho fg} \sigma^{fg})] - \\
&\frac{1}{8}\gamma_a g^{\mu\nu} \sum_{(\nu,b,c)} [\bar{\psi}_\nu \gamma_b (R^S(\eta,\xi)\psi_c) + (R^S(\eta,\xi)\bar{\psi}_\nu)\gamma_b \psi_c] \sigma^{bc} | \chi^n > \bar{\epsilon}\gamma^a \psi_\mu + \\
&\frac{1}{8} < \chi^n | i\gamma^a e^\rho_a \frac{\delta}{\delta \psi^\alpha_\mu} [\sum_{(\rho,b,c)} [\bar{\psi}_\rho \gamma_b \psi_c]] \sigma^{bc} | \chi^n > [(R(\eta,\xi)\partial_\mu)\epsilon^\alpha + \\
&\partial_\mu (R^S(\eta,\xi)\epsilon^\alpha) + \frac{1}{2} [\overset{\circ}{\omega}{}^{(5)}_{\mu abNS} (R(\eta,\xi))[\sigma^{ab}]^\alpha_\beta \epsilon^\beta + \overset{\circ}{\omega}{}^{(0)}_{\mu ab} [\sigma^{ab}]^\alpha_\beta (R^S(\eta,\xi)\epsilon^\beta)] + \\
&\frac{1}{8} [\sum_{(\mu,b,c)} [\bar{\psi}_\mu \gamma_b \psi_c][\sigma^{bc}]^\alpha_\beta (R^S(\eta,\xi)\epsilon^b) + \sum_{(\mu,b,c)} [\bar{\psi}_\mu \gamma_b (R^S(\eta,\xi)\psi_c) + \\
&(R^S(\eta,\xi)\bar{\psi}_\mu)\gamma_b \psi_c][\sigma^{bc}]^\alpha_\beta \epsilon^\beta] + < \chi^n | i\gamma^a \{ e^\rho_a \frac{\delta}{\delta \psi^\alpha_\mu} [\frac{1}{8} \sum_{(\rho,b,c)} [(R^S(\eta,\xi)\bar{\psi}_\rho)\gamma_b \psi_c + \\
&\bar{\psi}_\rho \gamma_b (R^S(\eta,\xi)\psi_c)] + [\delta^\mu_\nu \delta^\beta_\alpha (R(\eta,\xi) e^\rho_a) + \\
&\frac{\delta}{\delta \psi^\alpha_\mu} (R^S(\eta,\xi)\psi^\beta_\nu) e^\rho_a] \times \\
&\frac{\delta}{\delta \psi^\beta_\nu} [\frac{1}{8} \sum_{(\rho,b,c)} [\bar{\psi}_\rho \gamma_b \psi_c]] \} \sigma^{bc} | \chi^n > \mathcal{D}_\mu \epsilon^\alpha = 0
\end{aligned}
\tag{74}
$$

Equations (71), (73) and (74) can be interpreted as restrictions over the possible space-time manifolds. Specifically, they restrict the curvatures of the tangent bundle TM and of the spinor bundle SM. We note that these restrictions have been obtained for the case of two congruences on which two commuting vector fields ξ and η. Another severe assumption made in deducing the restrictions above was that the Dirac eigenspinors undergo parallel transport along the two vector fields. The most general cases are analyzed in[19].

The restrictions have not a nice form, even though they display some symmetry. Since they have such a complicate structure it is difficult to say how the curvatures should look for the manifold to permit, under the above hypothesis, the covariant transport of primary constraints on it. Nevertheless, the system of restrictions posses a trivial solution

$$
R(\eta,\xi) = R^S(\eta,\xi) = 0 \tag{75}
$$

which assures that at least manifolds with flat tangent bundle and also flat spinor bundle admits Dirac eigenspinors as global obsrvables.

VI. DISCUSSIONS AND CONCLUDING REMARKS

Throughout this paper we analyzed under what circumstances the eigenvalues of the Dirac operator can be used as observables of Euclidean supergravity. We saw that, as in the case of general relativity, the Dirac operator in the presence of local supersymmetry is an elliptic differential operator of first order. Then, on a compact (spin) manifold, it admits a discrete spectrum. However, its eigenspinors might not coordinate the covariant phase space of the theory which is composed by equivalence classes of gravitational supermultiplets through equations of motion. This might happen because the eigenvalues might fail to define bijective functions from the geometry to the set of infinite sequences, as is the case in nonsupersymmetric theory. If we insist that the theory should be described in terms of Dirac eigenvalues, we come to the crucial difference with respect to general relativity: the eigenvalues are no longer gauge invariant by construction. The gauge transformations of the present theory are spacetime diffeomorphisms, local rotations and $N = 1$ local supersymmetry. From the invariance of Dirac eigenvalues under gauge transformations, which is necessary in order to have them playing the role of dynamical variables, we obtain the equations (26), (29) and (33). These equations represent manifest constraints on the phase space. From them follow a second set of relations, namely (39), (46) and (47) which represent additional restrictions on the eigenspinors of the Dirac operator. To have a consistent theory, both sets of constraints must have nontrivial solutions. We also saw that these constraints play a major role not only in the classical theory, but also in the quantization process. Next we addressed the problem of global extension of these observables over the manifold M. To investigate this matter, we consider transporting the primary constraints from a given point to another one, the two being at the opposite corners of a rectangle made by four points that belong to the intersection of two congruences. Subject to some simplifying assumptions, we concluded that the curvature of the tangent bundle and that of the spinor bundle must satisfy the equations (71), (73 and (74). Even if these restrictions are expressed by rather complicated equations, it was shown that manifolds with flat tangent bundle and flat spinor bundle satisfy them. Therefore, on these manifolds, the Dirac eigenvalues can be promoted to global observables in the sense mentioned above. This is not surprisingly since flat manifolds seem to be related to supersymmetric theories.

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APPENDIX A

The symbol of an operator is an useful object in the study of the properties of differential operators defined on a manifold. These operators allow us to get relevant information about the manifold. These constructions are most frequently met in physics in the context of gauge theories where elliptic complexes and Atiyah-Singer index theorem are tools in use[15].

Let us locally define the symbol of a differential operator[17]. For the beginning, if A and B are vector spaces over $R(C)$, $U \subset R^n$ and $E = U \times A$, $F = U \times B$, then a *differential*

operator of order k is an $R(C)$-linear operator $D : \Gamma(E) \rightarrow \Gamma(F)$ such that for each n-tuple $\alpha = (\alpha_1, \cdots, \alpha_n)$ of nonnegative integers there exists an application $L_\alpha : U \rightarrow Hom(A, B)$ such that for all $f \subset \Gamma(E)$

$$D(f) = \sum_{|\alpha| \leq k} L_\alpha D^\alpha f, \tag{76}$$

where

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \quad , \quad |\alpha| = \sum_{i=1}^{n} \alpha_i. \tag{77}$$

Here $\Gamma(E)$ is the set of sections of E. Now let $y = (y_1, \cdots, y_n) \in R^n$ and $y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$ and $v = (x, y) \in U \times R^n$. Then there is an application $A \rightarrow B$ defined as follows

$$\sigma_v(D) = \sum_{|\alpha|} y^\alpha L_\alpha(x) \tag{78}$$

which defines a map from $U \times R^n \rightarrow Hom(A, B)$ called the *symbol* of D, denoted by $\sigma(D)$ and given by $v \rightarrow \sigma_v(D)$. We say that D is *elliptic* if $\sigma(D)$ is an isomorphism for all $v = (x, y)$ with $y \neq 0$.

The extension of these definitions to vector bundles is straightforward[17]. We note only that there exists a symbol of D defined as follows: for $x \in M$, $v \in T_x^* M$, $\sigma_v(D) : E_x \rightarrow F_x$ where E_x and F_x are the fibres over x of two vector-bundles E,F over the same smooth manifold M for which $\pi : T^*M \rightarrow M$ is the projection of the tangent bundle. Then for $e \in E_x$ and $s \in \Gamma(E)$ with $s(x) = e$ and $g \in C^\infty(M)$ with $g(x) = 0$ and $dg(x) = v$ the symbol of $D : \Gamma(E) \rightarrow \Gamma(F)$ is given by

$$\sigma_v(D)(e) = D(\frac{g^k}{k!} s)(x) \in F_x. \tag{79}$$

The map $v \rightarrow \sigma_v(D)$ defines an element $\sigma(D) \in \Gamma Hom(\pi^* E, \pi^* F)$ where $E_x \simeq (\pi^* E)_v$, $F_x \simeq (\pi^* F)_v$.

APPENDIX B

In this appendix we present the action of the two covariant derivatives on different objects. The minimal covariant derivative acting on a world vector $A = A^\mu \partial_\mu$ is given by

$$\nabla_\xi A^\nu = \xi^\mu (\partial_\mu A^\nu + \Gamma^\nu_{\mu\sigma} A^\sigma) = \xi^\mu \mathcal{D}_\mu A^\nu \tag{80}$$

and the covariant derivative of $SO(4)$ vectors $A = A^a \partial_a$ is given by

$$\nabla_\xi A^a = \xi^\mu (\partial_\mu A^a + \omega^a_{\mu b} A^b) = \xi^\mu \mathcal{D}_\mu A^a. \tag{81}$$

The covariant derivatives of the gravity supermultiplet are given by

$$\nabla_\xi e^a_\mu = \xi^\nu (\partial_\nu e^a_\mu - \Gamma^\rho_{\mu\nu} e^a_\rho + \omega^a_{\nu b}(e, \psi) e^b_\mu) = \xi^\nu \mathcal{D}_\nu e^a_\mu \tag{82}$$

for the graviton and

$$\nabla^S_\xi \psi^\alpha_\nu = \xi^\mu(\partial_\mu \psi^\alpha_\nu + \frac{1}{2}\omega_{\mu ab}(e,\psi)\sigma^{ab}\psi^\alpha_\nu) = \xi^\mu \mathcal{D}_\mu \psi^\alpha_\nu \tag{83}$$

for gravitino. The "supersymmetric spin connection" contains the usual spin connection and $K_{\mu ab}$ term depending on the gravitino

$$\omega_{\mu ab}(e,\psi) = \overset{\circ}{\omega}_{\mu ab}(e) + K_{\mu ab}(\psi). \tag{84}$$

APPENDIX C

Here are given some useful objects transported along path1:

$$\begin{aligned}
e^a_{\mu 1} &= e^{\mu\nabla_\eta} e^{\lambda\nabla_\xi} e^a_\mu \\
\psi^\alpha_{\mu 1} &= e^{\mu\nabla_\eta} e^{\lambda\nabla_\xi} \psi^\alpha_\mu \\
\partial_{\mu 1} &= e^{\mu\nabla_\eta} e^{\lambda\nabla_\xi} \partial_\mu \\
\frac{\delta}{\delta e^a_{\mu\, 1}} &= \frac{\delta(e^{-\lambda\nabla_\xi} e^{-\mu\nabla_\eta} e^b_\nu)}{\delta e^a_\mu} \frac{\delta}{\delta e^b_\nu} \\
\frac{\delta}{\delta \psi^\alpha_{\mu\, 1}} &= \frac{\delta(e^{-\lambda\nabla_\xi} e^{-\mu\nabla_\eta} \psi^\beta_\nu)}{\delta \psi^\alpha_\mu} \frac{\delta}{\delta \psi^\beta_\nu}
\end{aligned} \tag{85}$$

The essential coefficients of power expanding of the components of the primary constraints are given by

$$\mathcal{T}^{n\mu(0)}_{a1} =< \chi^n | i\gamma^d \delta_{da} \delta^{\mu\nu}(\partial_\nu + \frac{1}{2} \overset{\circ}{\omega}{}^{(0)}_{\nu fg} \sigma^{fg}) + \frac{i}{8}\gamma^d \delta_{da} g^{\mu\nu} \sum_{(\nu,b,c)} [\bar{\psi}_\nu \gamma_b \psi_c] \sigma^{bc} | \chi^n >$$

$$\begin{aligned}
\mathcal{T}^{n\mu(5)}_{a1NS} =< \chi^n | i\gamma^d \delta_{da} \delta^{\mu\nu} [(\nabla_\eta \nabla_\xi \partial_\nu) + \frac{1}{2} \overset{\circ}{\omega}{}^{(5)}_{\nu fgNS} (\nabla_\eta \nabla_\xi) \sigma^{fg}] + \\
i\gamma^d [\frac{\delta}{\delta e^a_\mu} (\nabla_\eta \nabla_\xi e^\rho_d)(\partial_\rho + \frac{1}{2} \overset{\circ}{\omega}{}^{(0)}_{\rho fg} \sigma^{fg}) + \frac{\delta}{\delta e^a_\mu} [(\nabla_\xi \nabla_\eta e_{\nu d}] \delta^{\nu\rho} (\partial_\rho + \\
\frac{1}{2} \overset{\circ}{\omega}{}^{(0)}_{\rho fg} \sigma^{fg})] + \frac{i}{8}\gamma^d \delta_{da} g^{\mu\nu} \sum_{(\nu,b,c)} [\bar{\psi}_\nu \gamma_b (\nabla^S_\eta \nabla^S_\xi \psi_c) + (\nabla^S_\eta \nabla^S_\xi \bar{\psi}_\nu) \gamma_b \psi_c] \sigma^{bc} | \chi^n >
\end{aligned}$$

$$\Gamma^{n\mu(0)}_{\alpha 1} =< \chi^n | \frac{i}{8} \gamma^a e^\rho_a \frac{\delta}{\delta \psi^\alpha_\mu} [\sum_{(\rho,b,c)} [\bar{\psi}_\rho \gamma_b \psi_c]] \sigma^{bc} | \chi^n >$$

$$\begin{aligned}
\Gamma^{n\mu(5)}_{\alpha 1NS} =< \chi^n | i\gamma^a \{ e^\rho_a \frac{\delta}{\delta \psi^\alpha_\mu} \frac{1}{8} [\sum_{(\rho,b,c)} [(\nabla^S_\eta \nabla^S_\xi \bar{\psi}_\rho) \gamma_b \psi_c + \bar{\psi}_\rho \gamma_b (\nabla^S_\eta \nabla^S_\xi \psi_c)] + [\delta^\mu_\nu \delta^\beta_\alpha (\nabla_\eta \nabla_\xi e^\rho_a) + \\
\frac{\delta}{\delta \psi^\alpha_\mu} (\nabla^S_\xi \nabla^S_\eta \psi^\beta_\nu) e^\rho_a] \frac{\delta}{\delta \psi^\beta_\nu} [\frac{1}{8} \sum_{(\rho,b,c)} [\bar{\psi}_\rho \gamma_b \psi_c]] \} \sigma^{bc} | \chi^n >
\end{aligned}$$

$$\partial_\nu e^{a(0)}_{\mu 1} = \partial_\nu e^a_\mu$$

$$\partial_\nu e^{a(5)}_{\mu 1NS} = (\nabla_\eta \nabla_\xi \partial_\nu) e^a_\mu + \partial_\nu (\nabla_\eta \nabla_\xi e^a_\mu)$$

$$\begin{aligned}
\partial_\nu \psi^{\alpha(0)}_{\mu 1} &= \partial_\nu \psi^\alpha_\mu \\
\partial_\nu \psi^{\alpha(5)}_{\mu 1NS} &= (\nabla^S_\eta \nabla^S_\xi \partial_\nu)\psi^\alpha_\mu + \partial_\nu(\nabla^S_\eta \nabla^S_\xi \psi^\alpha_\mu) \\
e^{(0)}_{b\mu 1} &= e^{(0)}_{b\mu 1} \\
e^{(5)}_{b\mu 1NS} &= \nabla_\eta \nabla_\xi e_{b\mu} \\
\psi^{\gamma(0)}_{\mu 1} &= \nabla^S_\eta \nabla^S_\xi \psi^\gamma_\mu \\
\psi^{\gamma(5)}_{\mu 1NS} &= \nabla^S_\eta \nabla^S_\xi \psi^\gamma_\mu \\
(\bar\epsilon \gamma^a \psi_\mu)^{(0)}_1 &= (\bar\epsilon \gamma^a \psi_\mu) \\
(\bar\epsilon \gamma^a \psi_\mu)^{(5)}_{1NS} &= [(\nabla^S_\eta \nabla^S_\xi \bar\epsilon)\gamma^a \psi_\mu + \bar\epsilon \gamma^a (\nabla^S_\eta \nabla^S_\xi \psi_\mu)] \\
(\mathcal{D}_\mu \epsilon^{\alpha(0)})_1 &= \mathcal{D}_\mu \epsilon^\alpha \\
(\mathcal{D}_\mu \epsilon^{\alpha(5)})_{1NS} &= (\nabla_\eta \nabla_\xi \partial_\mu)\epsilon^\alpha + \partial_\mu(\nabla^S_\eta \nabla^S_\xi \epsilon^\alpha) + \frac{1}{2}[\overset{\circ}{\omega}{}^{(5)}_{\mu abNS}\, [\sigma^{ab}]^\alpha_\beta \epsilon^\beta + \\
&\overset{\circ}{\omega}{}^{(0)}_{\mu ab}\, [\sigma^{ab}]^\alpha_\beta (\nabla^S_\eta \nabla^S_\xi \epsilon^\beta)] + \frac{1}{8}[\sum_{(\mu,b,c)} [\bar\psi_\mu \gamma_b \psi_c][\sigma^{bc}]^\alpha_\beta (\nabla^S_\eta \nabla^S_\xi \epsilon^b) + \\
&\sum_{(\mu,b,c)} [\bar\psi_\mu \gamma_b (\nabla^S_\eta \nabla^S_\xi \psi_c) + (\nabla^S_\eta \nabla^S_\xi \bar\psi_\mu)\gamma_b \psi_c][\sigma^{bc}]^\alpha_\beta \epsilon^\beta] \\
\overset{\circ}{\omega}{}^{(0)}_{\nu bc1} &= \overset{\circ}{\omega}_{\nu bc} \\
\overset{\circ}{\omega}{}^{(5)}_{\nu bc1NS} &= (\nabla_\eta \nabla_\xi e^\mu_a)\partial_{[\nu} e_{b\mu]} + \cdots + \frac{1}{2}(\nabla_\eta \nabla_\xi e^\rho_a) e^\sigma_b \partial_\sigma e_{\rho c} e^c_\nu + \frac{1}{2}(\nabla_\xi \nabla_\eta e^\rho_a) e^\sigma_b \partial_\sigma e_{\rho c} e^c_\nu + \cdots \\
K^{(0)}_{\nu ab1} &= K_{\nu ab} \\
K^{(5)}_{\nu ab1NS} &= \imath 8 \sum_{(\nu bc)} [\bar\psi_\nu \gamma_b (\nabla^S_\eta \nabla^S_\xi \psi_c) + (\nabla^S_\eta \nabla^S_\xi \bar\psi_\nu)\gamma_b \psi_c], \quad (86)
\end{aligned}$$

where $\partial_{[\nu} e_{b\mu]}$ refers to antisymmetrization with respect to the indices ν and μ only and the sums on fermion products in all the expressions used in this paper mean

$$\sum_{(\nu bc)} [A_\nu \gamma_b B_c] = A_\nu \gamma_b B_c - A_\nu \gamma_c B_b + A_b \gamma_\nu B_c. \quad (87)$$

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